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Functions of a complex variable in problems in the theory of elasticity with mass forces☆

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ABSTRACT

The general equations of the theory of elasticity are reduced to an inhomogeneous fourth-order equation assuming that there is a linear dependence of the third component of the displacement vector on the third coordinate and that a mass force potential exists. The solution of this equation is presented, in particular, using two complex Kolosov–Muskhelishvili potentials. A third complex potential is introduced in addition to these. Using the three complex potentials, expressions are obtained for the components of the displacement vector and the stress and strain tensors that take account of mass forces. The application of the three potentials is analysed in problems in the theory of elasticity, and analytical solutions of several plane strain problems are presented.

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The use of the theory of function of a complex variable (TFCV) in the theory of elasticity has led to the development of effective methods for solving plane strain and plane stress-state problems. Two complex Kolosov–Muskhelishvili potentials are used in these problems.[1,2] A third complex potential has recently been introduced[3] by means of which the possibility has emerged of improving the solutions of certain plane stress-state problems and of extending the domain of applicability of the TFCV (see for example, Refs4 and 5).

Insufficient attention has been paid to the use of TFCV methods in problems of the theory of elasticity when there are mass forces but this possibility has been discussed.[1,2,6] For example, the investigation of the stresses in a meteorite which is falling to Earth, the prediction of the deformability of components made of polymer materials during long-term storage or use, the analysis of the stresses in the rotating parts of structures, etc. are among these problems. A method for solving of plane problems in the theory of elasticity when there are mass forces is developed below using three complex potentials, and the solutions of several plane deformation problems are presented.

1. Representation of the components of the displacement vector and the stress and strain tensors using three complex potentials

It has been assumed earlier[3] that, in the deformation of thin plates of constant or variable thickness, the components of the displacement vector are given, in a rectangular Cartesian system of coordinates, in the form

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = g(x_1, x_2)x_3 \tag{1.1}$$

Here, the components of the strain ε_{ij} and stress σ_{ij} tensors are defined by the expressions

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = g$$
$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{13} = \frac{1}{2}\frac{\partial g}{\partial x_1}x_3, \quad \varepsilon_{23} = \frac{1}{2}\frac{\partial g}{\partial x_2}x_3 \tag{1.2}$$

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$$\sigma_{11} = \lambda\Theta + 2\mu\frac{\partial u_1}{\partial x_1}, \quad \sigma_{22} = \lambda\Theta + 2\mu\frac{\partial u_2}{\partial x_2}, \quad \sigma_{33} = \lambda\Theta + 2\mu g; \quad \Theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + g$$

$$\sigma_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \sigma_{13} = \mu\frac{\partial g}{\partial x_1}x_3, \quad \sigma_{23} = \mu\frac{\partial g}{\partial x_2}x_3 \tag{1.3}$$

where λ and μ are Lamé constants.

The above relations still hold in the case of the deformation of a body of cylindrical shape of infinite length if the relative displacements along the longitudinal axis of the cylinder depend linearly on x_3 or they are equal to zero.

We will specify the components of the mass force density vector $M = (m_1, m_2, m_3)$ in the equilibrium equations $\sigma_{ij,j} + m_j = 0$ in the form[2,6]

$$m_1 = -\frac{\partial V}{\partial x_1}, \quad m_2 = -\frac{\partial V}{\partial x_2}, \quad m_3 = 0 \tag{1.4}$$

where V is a certain potential.

Taking these assumptions into account, the equilibrium equations are written as follows:

$$\frac{\partial(\sigma_{11} + \mu g - V)}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial(\sigma_{22} + \mu g - V)}{\partial x_2} = 0,$$

$$\nabla_2^2 g = 0; \quad \nabla_2^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{1.5}$$

Eq. (1.5) are satisfied identically if we put

$$\sigma_{11} + \mu g - V = \frac{\partial^2 U}{\partial x_2^2}, \quad \sigma_{22} + \mu g - V = \frac{\partial^2 U}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 U}{\partial x_1 \partial x_2} \tag{1.6}$$

where U is a function of the stresses. It can be seen from the last equation of system (1.5) that $g(x_1, x_2)$ is a harmonic function. From relations (1.6), we have, in particular,

$$\sigma_{11} = \frac{\partial^2 U}{\partial x_2^2} - \mu g + V, \quad \sigma_{22} = \frac{\partial^2 U}{\partial x_1^2} - \mu g + V \tag{1.7}$$

The components of the strain tensor, defined using the Hooke's law relations

$$\varepsilon_{11} = \frac{\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})}{E}, \quad \varepsilon_{22} = \frac{\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})}{E}, \quad \varepsilon_{12} = \frac{(1+\nu)\sigma_{12}}{E} \tag{1.8}$$

where E is Young's modulus and ν is Poisson's ratio, are substituted into the compatibility condition

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2\varepsilon_{12}}{\partial x_1 \partial x_2}$$

As a result, we obtain

$$\frac{\partial^2}{\partial x_2^2}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] + \frac{\partial^2}{\partial x_1^2}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] = 2(1+\nu)\frac{\partial^2\sigma_{12}}{\partial x_1 \partial x_2} \tag{1.9}$$

Now, differentiating the first of the equilibrium Eq. (1.5) with respect to x_1 and the second with respect to x_2 and adding them together, we obtain

$$2\frac{\partial^2\sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2\sigma_{11}}{\partial x_1^2} - \frac{\partial^2\sigma_{22}}{\partial x_2^2} - \mu\nabla_2^2 g + \nabla_2^2 V$$

Substituting the last expression into Eq. (1.9) and taking account of the harmonicity of the function, we obtain

$$\nabla_2^2(\sigma_{11} + \sigma_{22}) - \nu\nabla_2^2\sigma_{33} = (1+\nu)\nabla_2^2 V \tag{1.10}$$

Using the relation $\sigma_{33} = Eg + \nu(\sigma_{11} + \sigma_{22})$, which follows from Hooke's law, we reduce Eq. (1.10) to the form

$$(1-\nu)\nabla_2^2(\sigma_{11} + \sigma_{22}) = \nabla_2^2 V$$

Substituting the sum $\sigma_{11} + \sigma_{22}$, which is determined using relations (1.7), into the last equation, we obtain

$$(1-\nu)\nabla_2^4 U = -(1-2\nu)\nabla_2^2 V \tag{1.11}$$

Assuming that U and V are functions of the complex variables $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ and taking account of the operator identity

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

we reduce Eq. (1.11) to the form

$$4(1-\nu)\frac{\partial^4 U}{\partial z^2 \partial \bar{z}^2} = -(1-2\nu)\frac{\partial^2 V}{\partial z \partial \bar{z}} \tag{1.12}$$

We will represent the general solution of the inhomogeneous Eq. (1.12) in the form of the sum of the general solution of the corresponding homogeneous equation and some particular solution of the inhomogeneous equation. We will represent the general solution of the homogeneous equation using Gourjats formula[2] and determine the particular solution by direct integration.

The general solution of Eq. (1.12) therefore has the form

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} + W \tag{1.13}$$

where $\varphi(z)$, $\chi(z)$ are complex potentials and

$$W = -\frac{1-2\nu}{2(1-\nu)}\int dz \int V d\bar{z} \tag{1.14}$$

Here,

$$\frac{\partial^2 W}{\partial z \partial \bar{z}} = -\frac{1-2\nu}{2(1-\nu)}V$$

It can be seen that, in accordance with relations (1.6),

$$\sigma_{11} + \sigma_{22} + 2\mu g - 2V = \nabla_2^2 U = 4\frac{\partial^2 U}{\partial z \partial \bar{z}}$$

In exactly the same way, we obtain the second combination, which will be used below

$$\sigma_{11} - \sigma_{22} + 2i\sigma_{12} = -4\frac{\partial^2 U}{\partial \bar{z}^2} \tag{1.15}$$

Taking account of expression (1.3), we have

$$\sigma_{11} + \sigma_{22} + 2\mu g - 2V = 2[\varphi'(z) + \overline{\varphi'(z)}] + 2\frac{\partial^2 W}{\partial z \partial \bar{z}} \tag{1.16}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\varphi''(z) + \chi''(z)] + 2\frac{\partial^2 W}{\partial z^2} \tag{1.17}$$

We now introduce the complex displacement $D = u_1 + iu_2$. It is easy to see that

$$2\frac{\partial D}{\partial \bar{z}} = \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + i\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \varepsilon_{11} - \varepsilon_{22} + 2i\varepsilon_{12} \tag{1.18}$$

Using Hooke's law, we obtain

$$4\mu\frac{\partial D}{\partial \bar{z}} = \sigma_{11} - \sigma_{22} + 2i\sigma_{12} \tag{1.19}$$

Comparing equalities (1.15) and (1.19), we have the equation

$$4\mu\frac{\partial D}{\partial \bar{z}} = -4\frac{\partial^2 U}{\partial \bar{z}^2}$$

which is satisfied if we put

$$4\mu D = f(z) - 4\frac{\partial U}{\partial \bar{z}}$$

where $f(z)$ is now considered as a third complex potential in addition to $\varphi(z)$, $\chi(z)$. Using it, the representation of the components of the displacement vector in terms of the three complex potentials takes the form

$$4\mu D = 4\mu(u_1 + iu_2) = f(z) - 2[\varphi(z) + z\overline{\varphi'(z)} + \overline{\chi'(z)}] - 2\frac{\partial W}{\partial \bar{z}}$$

On now differentiating this equality with respect to z, we obtain

$$4\mu\frac{\partial D}{\partial z} = 2\mu\left[\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)-i\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right)\right] = f'(z)-2[\varphi'(z)+\overline{\varphi'(z)}]-2\frac{\partial^2 W}{\partial z\partial\bar{z}}$$

Separating out the real terms in the last expression and introducing the notation

$$\{\varphi\} = \varphi'(z)+\overline{\varphi'(z)},\quad \{f\} = f'(z)+\overline{f'(z)},\quad G(z,\bar{z}) = \frac{\partial^2 W}{\partial z\partial\bar{z}},\quad H(z,\bar{z}) = \frac{\partial^2 W}{\partial z^2} \tag{1.20}$$

we have

$$4\mu(\varepsilon_{11}+\varepsilon_{22}) = \{f\}-4\{\varphi\}-4G(z,\bar{z}) \tag{1.21}$$

In order to represent the function $g(z,\bar{z})$ using the above three complex potentials, we turn to Hooke's law whence we have

$$\sigma_{\beta\beta} = \lambda(\varepsilon_{11}+\varepsilon_{22}+g)+2\mu\varepsilon_{\beta\beta},\quad \beta = 1, 2$$

Then,

$$\sigma_{11}+\sigma_{22} = 2(\lambda+\mu)(\varepsilon_{11}+\varepsilon_{22})+2\lambda g \tag{1.22}$$

The quantity $2\,\mu g$ is added to both sides of equality (1.22), after which this equality is represented in the form

$$g = \frac{1}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22}+2\mu g)-(\varepsilon_{11}+\varepsilon_{22}) \tag{1.23}$$

The quantity $\sigma_{11}+\sigma_{22}+2\mu g$ is determined from equality (1.6), taking account of expression (1.14).
Then,

$$\sigma_{11}+\sigma_{22}+2\mu g = 2\{\varphi\}+\frac{1}{1-\nu}V \tag{1.24}$$

Taking account of equalities (1.21) and (1.24), as well as (1.14), we reduce relation (1.23) to the form

$$g(z,\bar{z}) = \frac{\lambda+2\mu}{\mu(\lambda+\mu)}\{\varphi\}-\frac{1}{4\mu}\{f\} \tag{1.25}$$

The resulting equality (1.25) enables us to modernize the expression for the sum $\sigma_{11}+\sigma_{22}$, for which we make use of relation (1.24). We then obtain

$$\sigma_{11}+\sigma_{22} = \frac{1}{2}\{f\}-\frac{2\mu}{\lambda+\mu}\{\varphi\}+\frac{1}{1-\nu}V$$

The component σ_{33} of the stress tensor is determined using the relation

$$\sigma_{33} = \lambda(\varepsilon_{11}+\varepsilon_{22}+g)+2\mu g = \lambda(\varepsilon_{11}+\varepsilon_{22})+(\lambda+2\mu)g$$

Using equalities (1.21) and (1.25), we obtain an explicit expression for σ_{33} (see below).
It is also easily seen that the complex quantity $S=\sigma_{13}-i\sigma_{23}$ is defined by the expression

$$\sigma_{13}-i\sigma_{23} = \mu x_3\left(\frac{\partial g}{\partial x_1}-i\frac{\partial g}{\partial x_2}\right) = 2\mu x_3\frac{\partial g}{\partial z}$$

Using expression (1.25), we obtain

$$\sigma_{13}-i\sigma_{23} = 2x_3\left[\frac{\lambda+2\mu}{\lambda+\mu}\phi''(z)-\frac{1}{4}f''(z)\right]$$

and the separate components σ_{13} and σ_{23} are easily found from this.

2. The use of three complex potentials in boundary value problems in the theory of elasticity

Starting from the relations obtained in the preceding section and introducing, in addition to (1.20), the notation

$$\{\phi\} = \bar{z}\varphi''(z)+\phi'(z),\quad \{\varphi^{\pm}\} = \varphi''(z)\pm\overline{\varphi''(z)}$$

$$\{f^{\pm}\} = f''(z)\pm\overline{f''(z)},\quad J(z,\bar{z}) = \frac{\partial^2 W}{\partial\bar{z}^2}$$

we now present expressions for the components of the stress tensor in a form which will later be more convenient

$$\sigma_{11} = \frac{1}{4}\{f\} - (1-2\nu)\{\varphi\} - \frac{1}{2}\{\phi\} - \frac{1}{2}\{\bar{\phi}\} - \frac{H(z,\bar{z}) + J(z,\bar{z})}{2} - \frac{1}{1-2\nu}G(z,\bar{z})$$

$$\sigma_{22} = \frac{1}{4}\{f\} - (1-2\nu)\{\varphi\} + \frac{1}{2}\{\phi\} + \frac{1}{2}\{\bar{\phi}\} + \frac{H(z,\bar{z}) + J(z,\bar{z})}{2} - \frac{1}{1-2\nu}G(z,\bar{z})$$

$$\sigma_{33} = 2(2-\nu)\{\varphi\} - \frac{1}{2}\{f\} - \frac{2\nu}{1-2\nu}G(z,\bar{z})$$

$$\sigma_{12} = -\frac{i}{2}[\{\phi\} - \{\bar{\phi}\} + H(z,\bar{z}) - J(z,\bar{z})]$$

$$\sigma_{13} = x_3\left[2(1-\nu)\{\varphi^+\} - \frac{1}{4}\{f^+\}\right], \quad \sigma_{23} = ix_3\left[2(1-\nu)\{\varphi^-\} - \frac{1}{4}\{f^-\}\right] \tag{2.1}$$

It can be directly verified that the components of the stresses (2.1) identically satisfy the equilibrium equations and the Beltrami–Mitchell compatibility conditions.

As a rule, expressions (2.1) are very cumbersome and, in practical applications of TFCV methods in the theory of elasticity, it is customary to start out from specific combinations of the required components of the stress or strain tensors, force vectors or displacements Without setting ourselves the problem of giving a complete review of these, we will present several combinations using the components of the stress tensor which are most important for the subsequent treatment. They are:

$$\sigma_{11} + \sigma_{22} = \frac{1}{2}\{f\} - 2(1-2\nu)\{\varphi\} - \frac{2}{1-2\nu}G(z,\bar{z})$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\{\phi\} + 2H(z,\bar{z}) \tag{2.2}$$

The presence of the third complex potential somewhat complicates the procedure for solving of boundary value problems in the theory of elasticity. However, in the simplest cases (in plane strain problems and plane problems of the deformation of thin plates of constant thickness), this fact does not lead to any marked complications. Precisely, by putting

$$\varepsilon_{33} = \frac{1}{\mu}\left[2(1-\nu)\{\varphi\} - \frac{1}{4}\{f\}\right] = 0 \tag{2.3}$$

in the first case, we obtain

$$\{f\} = 8(1-\nu)\{\varphi\}$$

At the same time, the first relation of (2.2) is simplified and takes the form

$$\sigma_1 + \sigma_{22} = 2\{\varphi\} - \frac{2}{1-2\nu}G(z,\bar{z}) \tag{2.4}$$

Assumption (2.3) has no effect on the form of the second relation of (2.2).

In the case of the deformation of thin plates on a face, the normal stress $\sigma_{33} = 0$. In this case, the first of relations (2.2) takes the form

$$\sigma_{11} + \sigma_{22} = 2(1+\nu)\{\varphi\} - \frac{2(1+\nu)}{1-2\nu}G(z,\bar{z}) \tag{2.5}$$

With this assumption, the form of the second relation of (2.2) also remains unchanged.

In the classical case,[2] instead of relations (2.4) and (2.5), we have

$$\sigma_{11} + \sigma_{22} = 2\{\varphi\}$$

Note that the use of three complex potentials enables us to obtain the exact solutions of plane stress-state problems[7] instead of approximate solutions. However, at the same time, shear stresses associated with the third coordinate can emerge. By virtue of their linear dependence on the third coordinate, they satisfy the conditions for static equilibrium in an integral sense.[4]

In many cases, conformal mappings of specified domains in the z plane into a domain of the ζ plane in the form of a circle, a circular ring or an infinite plane with a circular aperture are used in the implementation of the methods of the theory of functions of a complex variable.[2] In this case, the polar coordinates ρ and υ are introduced in the ζ plane, where $\zeta = \rho e^{i\upsilon}$. We will determine the form of relations (2.2) in this system of coordinates. The following notation is introduced

$$F(z) = f'(z), \quad \Phi(z) = \varphi'(z), \quad \Psi(z) = \phi'(z)$$

We shall assume that a conformal mapping

$$z = \omega(\varsigma) \tag{2.6}$$

is given and we will use the notation

$$\Phi_1(\varsigma) = \Phi(z) = \Phi[\omega(\varsigma)], \ldots, \quad G_1(\varsigma,\bar{\varsigma}) = G(z,\bar{z}) = G[\omega(\varsigma), \overline{\omega(\varsigma)}], \ldots$$

Then,

$$\Phi'(z) = \Phi_1'(\varsigma)/\omega'(\varsigma)$$

Hence, when transformation (2.6) is carried out, relations (2.2) take the form

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} = \frac{1}{2}[F_1(\varsigma) + \overline{F_1(\varsigma)}] - 2(1-2\nu)[\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{2}{1-2\nu}G_1(\varsigma, \bar{\varsigma}) \tag{2.7}$$

$$\sigma_{\vartheta\vartheta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\vartheta} = \frac{2\varsigma^2\omega'(\varsigma)}{\rho^2\overline{\omega'(\varsigma)}}\left\{\left[\frac{\overline{\omega(\varsigma)}}{\omega'(\varsigma)}\Phi_1'(\varsigma) + \Psi_1(\varsigma)\right] + H_1(\varsigma, \bar{\varsigma})\right\} \tag{2.8}$$

Relation (2.7) is simplified in plane strain problems ($\varepsilon_{33} = 0$):

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} = 2[\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{2}{1-2\nu}G_1(\varsigma, \bar{\varsigma}) \tag{2.9}$$

It is well known that, in order to obtain the solution of a problem in the theory of elasticity, it is also necessary to require that boundary conditions are satisfied. At the same time, it is convenient to use particular combinations made up of complex potentials. In plane strain problems, such a combination can be obtained by subtracting equality (2.8) from equality (2.9). As a result, we obtain the formula

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta} = [\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{1}{1-2\nu}G_1(\varsigma, \bar{\varsigma}) -$$

$$- \frac{\varsigma^2\omega'(\varsigma)}{\rho^2\overline{\omega'(\varsigma)}}\left\{\left[\frac{\overline{\omega(\varsigma)}}{\omega'(\varsigma)}\Phi_1'(\varsigma) + \Psi_1(\varsigma)\right] + H_1(\varsigma, \bar{\varsigma})\right\} \tag{2.10}$$

which is widely used when implementing boundary conditions on a contour $\rho = \text{const}$.[2]

3. Stresses in rotating cylinders

In problems of the rotation of cylinders, we take the mass force potential in the form

$$V = -\frac{C\Omega^2 z\bar{z}}{2} \tag{3.1}$$

where C is the density of the material and Ω is the angular velocity of rotation of the cylinder. In this case, the particular solution of the inhomogeneous Eq. (1.12) is

$$W = \frac{1-2\nu}{16(1-\nu)}C\Omega^2 z^2\bar{z}^2 \tag{3.2}$$

At the same time,

$$G(z, \bar{z}) = \frac{\partial^2 W}{\partial z \partial \bar{z}} = \frac{1-2\nu}{4(1-\nu)}C\Omega^2 z\bar{z}, \quad H(z, \bar{z}) = \frac{\partial^2 W}{\partial z^2} = \frac{1-2\nu}{8(1-\nu)}C\Omega^2\bar{z}^2 \tag{3.3}$$

Since $z\bar{z} = x_1^2 + x_2^2$, then

$$m_1 = -\frac{\partial V}{\partial x_1} = C\Omega^2 x_1, \quad m_2 = -\frac{\partial V}{\partial x_2} = C\Omega^2 x_2$$

In a polar system of coordinates, the component of the mass force vector in the radial direction per unit volume has the form $C\Omega^2 r$. Note that the above expressions for the components of the mass force vector are identical with the representations given in the literature.[7]

3.1. Rotation of a circular cylinder

We will assume that an infinitely long, circular cylinder of radius R rotates about its axis with an angular velocity Ω. The outer surface of the cylinder is assumed to be stress-free.

In this case, we use the conformal mapping

$$z = R\varsigma \tag{3.4}$$

Then,

$$G_1(\varsigma, \bar{\varsigma}) = \frac{1-2\nu}{4(1-\nu)}K\varsigma\bar{\varsigma}, \quad H_1(\varsigma, \bar{\varsigma}) = \frac{1-2\nu}{8(1-\nu)}K\bar{\varsigma}^2; \quad K = C\Omega^2 R^2$$

and expression (2.10) takes the form

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta} = [\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{1}{4(1-\nu)}K\varsigma\bar{\varsigma} -$$

$$-\frac{\varsigma^2\omega'(\varsigma)}{\rho^2\overline{\omega'(\varsigma)}}\left\{\left[\frac{\overline{\omega(\varsigma)}}{\omega'(\varsigma)}\Phi_1'(\varsigma) + \Psi_1(\varsigma)\right] + \frac{1-2}{8(1-\nu)}K\bar{\varsigma}^2\right\} \tag{3.5}$$

Taking account of the equality (3.4), we write the boundary condition on the contour γ: $|\zeta| = 1$

$$[\Phi_1(t) + \overline{\Phi_1(t)}] - t\Phi_1'(t) - t^2\Psi_1(t) = \frac{3-2\nu}{8(1-\nu)}K \tag{3.6}$$

where $t = e^{i\upsilon}$ is an arbitrary point of the contour γ.

We will seek the complex potentials in the form

$$\Phi_1(\varsigma) = \sum_{k=0}^{\infty} a_k\varsigma^k, \quad \Psi_1(\varsigma) = \sum_{k=0}^{\infty} b_k\varsigma^k \tag{3.7}$$

Substituting these expressions into equality (3.6), we find

$$a_0 + \bar{a}_0 = \frac{3-2\nu}{8(1-\nu)}K$$

Only the real part a_{01} of the coefficient a_o:

$$a_{01} = \frac{3-2\nu}{16(1-\nu)}K$$

is determined from this. The imaginary part a_{02} of the coefficient a_0 remains uncertain, but since it does not play any role in this case it can be put equal to zero.

It is easily seen that all the remaining coefficients b_0, a_k, b_k $(k \geq 1)$ vanish and, therefore,

$$\Phi_1(\varsigma) = \frac{3-2\nu}{16(1-\nu)}K, \quad \Psi_1(\varsigma) = 0$$

Using the expressions obtained for the complex potentials and equalities (2.8) and (2.9), we obtain

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} = \frac{3-2\nu}{4(1-\nu)}K - \frac{1}{2(1-\nu)}K\rho^2, \quad \sigma_{\vartheta\vartheta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\vartheta} = \frac{1-2\nu}{4(1-\nu)}K\rho^2$$

Solving this system, we obtain

$$\sigma_{\rho\rho} = \frac{3-2\nu}{4(1-\nu)}K(1-\rho^2), \quad \sigma_{\vartheta\vartheta} = \frac{3-2\nu}{8(1-\nu)}K(1-\rho^2) + \frac{1-2\nu}{4(1-\nu)}K\rho^2, \quad \sigma_{\rho\vartheta} = 0$$

In this case, the expression for the component σ_{33} of the stress tensor has the form

$$\sigma_{33} = 2\nu[\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{2\nu}{1-2\nu}G_1(\varsigma, \bar{\varsigma}) = \frac{\nu K}{4(1-\nu)}(3 - 2\nu - 2\rho^2)$$

Using the equality $\rho = r/R$ and changing to the initial variable we arrive at the well-known solution.[7]

3.2. Eccentric rotation of a circular cylinder

We will now consider an infinitely long circular cylinder of radius R which rotates about a line parallel to its axis with an angular velocity Ω. We will assume that this line in the X_1OX_2 plane passes through the origin of coordinates and is separated from the axis of the cylinder by a distance $R|a|$, $0 \leq |a| < 1$. We shall assume that the outer surface of the cylinder is completely stress-free.

In the case considered, we will use the superposition of the two successive mappings

$$\varsigma_1(\varsigma) = \frac{a+\varsigma}{1+a\varsigma}, \quad z = R(\varsigma_1 - a)$$

as the conformal mapping. The first of these maps a circle of unit radius in the ζ plane into a unit circle in the ζ_1 plane. Here, the point $\zeta = 0$ becomes into the point $\zeta_1 = a$ and the points $\zeta = \pm 1$ become the points $\zeta_1 = \pm 1$ respectively. The second function performs a displacement and an extension. As a result, the interior of a circle of unit radius is mapped into the interior of a circle of radius R in the z plane and the above-mentioned points become, in the same sequence respectively, the points

$$z = 0, \quad z = R(1-a), \quad z = -R(1+a)$$

Taking $a = -0.6$ as an example, we obtain that the initial cylinder of radius R rotates about a line parallel to the longitudinal axis of the cylinder and this line, the axis of rotation, is separated from the axis of the cylinder by a distance $0.6R$. Without loss of generality, we shall assume that a is a real number.

We therefore give the conformal mapping in the form

$$z = \omega(\varsigma) = R\frac{(1-a^2)\varsigma}{1+a\varsigma} \tag{3.8}$$

Here,

$$\omega'(\varsigma) = R\frac{1-a^2}{(1+a\varsigma)^2}$$

Taking account of equality (3.8), we write relations (3.2) and (3.3) in the form

$$G_1(\varsigma,\bar{\varsigma}) = \frac{1-2\nu}{4(1-\nu)}\frac{(1-a^2)K\varsigma\bar{\varsigma}}{(1+a\varsigma)(1+a\bar{\varsigma})}, \quad H_1(\varsigma,\bar{\varsigma}) = \frac{1-2\nu}{8(1-\nu)}\frac{(1-a^2)K\bar{\varsigma}^2}{(1+a\bar{\varsigma})^2}$$

As a result, the basic combinations for the components of the stress tensor, taking account of plane strain $\varepsilon_{33} = 0$, take the form

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} = 2[\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{1}{2(1-\nu)}\frac{(1-a)K\varsigma\bar{\varsigma}}{(1+a\varsigma)(1+a\bar{\varsigma})}$$

$$\sigma_{\vartheta\vartheta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\vartheta} = \frac{2\varsigma^2\omega'(\varsigma)}{\rho^2\overline{\omega'(\varsigma)}}\left\{\left[\frac{\overline{\omega(\varsigma)}}{\omega'(\varsigma)}\Phi_1'(\varsigma) + \Psi_1(\varsigma)\right] + \frac{1-2\nu}{8(1-\nu)}\frac{(1-a^2)K\bar{\varsigma}^2}{(1+a\bar{\varsigma})^2}\right\} \tag{3.9}$$

Using relations (3.9), we form the combination used in the boundary condition which, for the problem considered, on the contour γ: $|\zeta| = 1$ of the unit circle takes the form

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta}|_\gamma = 0 \tag{3.10}$$

Using relations (3.9), we write the above-mentioned boundary condition, after reduction to a common denominator, as follows:

$$\begin{aligned}&[\Phi_1(t) + \overline{\Phi_1(t)}](t+a)(1+at)^2 - (t+a)^2(1+at)^2\Phi_1'(t) - (t+a)^3\Psi_1(t) = \\ &= \frac{1}{4(1-\nu)}K(1-a^2)^2t(1+at) + \frac{1-2\nu}{8(1-\nu)}K(1-a^2)^2(t+a)\end{aligned} \tag{3.11}$$

In this case, we also specify the complex potentials in the form (3.7). After substituting them into equality (3.11), by choosing the coefficients of like powers of t^k we obtain a system of equations in the unknown coefficients. For example, when $k \le -1$, the system of such equations takes the form

$$\begin{aligned}&a\bar{a}_1 + (2a^2+1)\bar{a}^2 + (a^3+2a)\bar{a}_3 + a^2\bar{a}_4 = 0\\ &a\bar{a}_2 + (2a^2+1)\bar{a}_3 + (a^3+2a)\bar{a}_4 + a^2\bar{a}_5 = 0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &a\bar{a}_k + (2a^2+1)\bar{a}_{k+1} + (a^3+2a)\bar{a}_{k+2} + a^2\bar{a}_{k+3} = 0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Putting $\bar{a}_{k+1} = \bar{a}_{k+2} = \bar{a}_{k+3} = 0$ for sufficiently large values of k, we successively obtain $a_k = a_{k-1} = \ldots = a_2 = a_1 = 0$. We conclude from this that

$$\Phi_1(\varsigma) = a_0 \tag{3.12}$$

Since an exact expression is obtained for the first complex potential, it is convenient, for determining the second complex potential, to apply a Cauchy-type integral to relation (3.11), both sides of which only contain functions which are holomorphic within the contour γ as terms. Carrying out this operation, we obtain

$$\begin{aligned}&(a_0 + \bar{a}_0)(\varsigma+a)(1+a\varsigma)^2 - \Psi_1(\varsigma)(\varsigma+a)^3 = \\ &= \frac{1}{4(1-\nu)}(1-a^2)^2K\varsigma(1+a\varsigma) + \frac{1-2\nu}{8(1-\nu)}(1-a^2)^2K(\varsigma+a)\end{aligned}$$

Hence, we find

$$\Psi_1(\varsigma) = (a_0 + \bar{a}_0)\frac{(1+a\varsigma)^2}{(\varsigma+a)^2} - \frac{1}{4(1-\nu)}\frac{(1-a^2)^2K\varsigma(1+a\varsigma)}{(\varsigma+a)^3} - \frac{1-2\nu}{8(1-\nu)}\frac{(1-a^2)^2K}{(\varsigma+a)^2} \tag{3.13}$$

We will use the following fact in order to determine the coefficient a_0. It follows from equality (3.12) that the complex potential $\Phi_1(\zeta)$ is independent of the value of the parameter a, which varies in the semi-interval [0,1). Hence, putting $a = 0$, we make use of the equality

$$\Psi_1(\varsigma) = 0$$

obtained above for this case by virtue of which, from formula (3.13), we find that

$$a_0 + \bar{a}_0 = \frac{3-2\nu}{8(1-\nu)}K$$

Hence, only the real part a_0 of the coefficient a_0:

$$a_{01} = \frac{3-2\nu}{16(1-\nu)}K$$

is determined. In the same way as above, the imaginary part a_{02} of the coefficient a_0 remains uncertain and we put it equal to zero.

Using the expressions obtained for the complex potentials, and also (2.9) and (2.8) respectively, we have

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} = \frac{3-2\nu}{4(1-\nu)}K - \frac{1}{2(1-\nu)}\frac{(1-a)K\varsigma\bar{\varsigma}}{(1+a\varsigma)(1+a\bar{\varsigma})} \tag{3.14}$$

$$\sigma_{\vartheta\vartheta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\vartheta} = \frac{2\varsigma^2(1+a\varsigma)^2}{\rho^2(1+a\bar{\varsigma})^2}\left\{\frac{K}{8(1-\nu)(\varsigma+a)^3}[(3-2\nu)(\varsigma+a)(1+a\varsigma)^2 - \right.$$

$$\left. -(1-2\nu)(1-a^2)^2(\varsigma+a) - 2\varsigma(1-a^2)^2(1+a\varsigma)] + \frac{1-2\nu}{8(1-\nu)}\frac{(1-a^2)K\bar{\varsigma}^2}{(1+a\bar{\varsigma})^2}\right\} \tag{3.15}$$

It is easy to verify that the boundary conditions are satisfied, especially on the axis of symmetry of a cross-section, and also the fact that, when $a = 0$, the combination of the components of the stress tensor obtained are identical to the analogous expressions obtained above for a cylinder rotating about its central axis. Using the formula for the inverse change of variables

$$\varsigma = \frac{z}{R(1-a^2) - az}$$

relations (3.14) and (3.15) can be expressed in terms of the coordinates of the z plane.

3.3. Rotation of an elliptic cylinder

We will now consider an infinite cylinder of elliptic cross-section. We shall assume that the semi-major axis of the ellipse is equal to a and the semi-minor axis is equal to b and the cylinder rotates about its longitudinal axis at an angular velocity Ω. We shall also assume that no forces of any kind are applied to the outer surface of the cylinder.

We shall determine the stresses arising in an elliptic cylinder as a result of its rotation using the conformal mapping

$$z = R\left(\frac{1}{\varsigma} + m\varsigma\right), \quad R = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b}$$

which maps an infinite plane $\zeta(\zeta = \rho e^{i\upsilon})$ with a circular aperture of unit radius $|\rho| = 1$ into the above-mentioned ellipse. Here, we use the notation

$$\Phi_1(\varsigma) = \Phi[\omega(\varsigma)], \quad \Psi_1(\varsigma) = \Psi[\omega(\varsigma)]$$

and derive specific expressions for the functions $G_1(\zeta, \bar{\zeta})$, $H_1(\zeta, \bar{\zeta})$:

$$G_1(\varsigma, \bar{\varsigma}) = \frac{K}{4(1-\nu)}\left(\frac{1}{\varsigma} + m\varsigma\right)\left(\frac{1}{\bar{\varsigma}} + m\bar{\varsigma}\right), \quad H_1(\varsigma, \bar{\varsigma}) = \frac{1-2\nu}{8(1-\nu)}K\left(\frac{1}{\bar{\varsigma}} + m\bar{\varsigma}\right)^2$$

Since no forces of any kind are applied to the outer surface of the cylinder, it is obvious that we have the boundary condition

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta}|_\gamma = 0$$

on the contour γ: $|\rho| = 1$. Omitting the transformations, we present the relation which expresses the given boundary condition:

$$(m-t^2)[\Phi_1(t)+\overline{\Phi_1(t)}]-(t^3+mt)\Phi_1'(t)-(mt^2-1)\Psi_1(t) = \\ = \frac{K}{4(1-\nu)}\left(m^3+\frac{m^2}{t^2}-t^2-mt^4\right)+ \\ +\frac{1-2\nu}{8(1-\nu)}K\left[(m^3-2m)-\frac{m^2}{t^2}+(2m^2-1)t^2+mt^4\right] \tag{3.16}$$

where $t = e^{i\upsilon}$ is an arbitrary point of the contour γ.

In the case being considered, we take the complex potentials in the form

$$\Phi_1(\varsigma) = \sum_{k=0}^{\infty} A_k\varsigma^{-k}, \quad \Psi_1(\varsigma) = \sum_{k=0}^{\infty} B_k\varsigma^{-k} \tag{3.17}$$

We substitute expressions (3.17) into equality (3.16) and equate the coefficients of like powers of t. We now consider the fact that all such equations in the coefficients with odd indices will be homogeneous. Hence, taking account of the uniqueness of the displacements,[3] from which it follows that $A_1 = B_1 = 0$, we conclude that all such coefficients are equal to zero.

Furthermore, for all $k > 2$, we have

$$mA_{k+2}-A_k = 0 \tag{3.18}$$

Hence, we conclude that, under the condition that the expansions presented converge, $A_3, A_4, A_5, \ldots$ are equal to zero. It is exactly the same in the case of negative exponents of t and, when $k < -2$ and account is taken of equality (3.18), we have $-mB_{k+2} + B_k = 0$. On the same grounds as above, we obtain that $B3, B4, B5, \ldots$ are equal to zero. Hence, we have the system of equations

$$m(A_0+\bar{A}_0)+A_2+B_0-mB_2 = \frac{3-2\nu}{8(1-\nu)}m^3K-\frac{1-2\nu}{4(1-\nu)}mK$$

$$-(A_0+\bar{A}_0)+mA_2-B_0 = -\frac{3-2\nu}{8(1-\nu)}K+\frac{1-2\nu}{4(1-\nu)}m^2K$$

$$3mA_2+B_2 = \frac{1+2\nu}{8(1-\nu)}m^2K$$

$$-A_2 = -\frac{1+2\nu}{8(1-\nu)}mK$$

in the unknowns $A_0, \bar{A}_0, A_2B_0, B_2$. Note that the imaginary part of the coefficient A_0 has no effect on the stress distribution and hence it can be put equal to zero. Taking account of this remark, we find

$$A_0 = \frac{3-2\nu}{16(1-\nu)}\frac{1+m^2}{1-m^2}K-\frac{1-6\nu}{16(1-\nu)}\frac{m^2(1+m^2)}{1-m^2}K, \quad A_2 = \frac{1+2\nu}{8(1-\nu)}mK$$

$$B_0 = \frac{1-6\nu}{4(1-\nu)}\frac{m^3}{1-m^2}K-\frac{3-2\nu}{4(1-\nu)}\frac{m}{1-m^2}K, \quad B_2 = -\frac{1+2\nu}{4(1-\nu)}m^2K$$

We therefore have the following expressions for the complex potentials

$$\Phi_1(\varsigma) = \left[\frac{3-2\nu}{16(1-\nu)}\frac{1+m^2}{1-m^2}-\frac{1-6\nu}{16(1-\nu)}\frac{m^2(1+m^2)}{1-m^2}\right]K+\frac{1+2\nu}{8(1-\nu)}mK\frac{1}{\varsigma^2}$$

$$\Psi_1(\varsigma) = \left[\frac{1-6\nu}{4(1-\nu)}\frac{m^3}{1-m^2}-\frac{3-2\nu}{4(1-\nu)}\frac{m}{1-m^2}\right]K-\frac{1+2\nu}{4(1-\nu)}m^2K\frac{1}{\varsigma^2}$$

It is easily seen that, when $m = 0$, the complex potentials are respectively identical to the complex potentials in the problem of the rotation of a circular cylinder.

The expressions

$$\sigma_{\rho\rho}+\sigma_{\vartheta\vartheta} = 2[\Phi_1(\varsigma)+\overline{\Phi_1(\varsigma)}]-\frac{K}{2(1-\nu)}\left(\frac{1}{\varsigma}+m\varsigma\right)\left(\frac{1}{\bar{\varsigma}}+m\bar{\varsigma}\right)$$

$$\sigma_{\vartheta\vartheta}-\sigma_{\rho\rho}+2i\sigma_{\rho\vartheta} = 2\frac{\bar{\varsigma}^2}{\rho^2}\frac{m\varsigma^2-1}{m\bar{\varsigma}^2-1}\left\{\frac{1+m\bar{\varsigma}^2}{\bar{\varsigma}(m\varsigma^2-1)}\Phi_1'(\varsigma)+\right.$$

$$\left.+\Psi_1(\varsigma)+\frac{1-2\nu}{8(1-\nu)}K\left(\frac{1}{\bar{\varsigma}}+m\bar{\varsigma}\right)^2\right\}$$

are used directly to determine the components of the stress tensor.

Separating the real and imaginary parts in them, we successively find expressions for the components of the stress tensor $\sigma_{\rho\rho}$, $\sigma_{\upsilon\upsilon}$, $\sigma_{\rho\upsilon}$. We find the σ_{33} component of the stress tensor from the condition $\varepsilon_{33}=0$:

$$\sigma_{33} = \nu(\sigma_{\rho\rho}+\sigma_{\vartheta\vartheta})$$

It can be established by direct verification that the components $\sigma_{\rho\rho}$, $\sigma_{\rho\upsilon}$ obtained in this way vanish on the contour of the aperture γ, that is, the expressions for the components of the stress tensor give the solution of the problem.

As a small illustration, we will present the relative numerical values of the tangential stresses $\sigma_{\upsilon\upsilon}^{(a)}$ and $\sigma_{\upsilon\upsilon}^{(b)}$ on the end of the semi-major and semi-minor axes respectively in the case of a constant length of the semi-major axis $a=10$ and a varying length of the semi-minor axis b:

b	10	5	4	3	2	1
$\sigma_{\vartheta\vartheta}^{(a)}$	1250	1289	1831	2894	5250	12800
$\sigma_{\vartheta\vartheta}^{(b)}$	1250	2227	2881	4032	6450	14030

4. Stresses in cylinders of infinite length under the action of mass forces in a transverse direction

We will assume that an infinitely long circular cylinder of radius R lies on a smooth plane. By virtue of the pliability of the materials of the cylinder and the base, we shall assume that a certain strip along the generatrix of the cylinder makes actual contact which enables us to avoid difficulties in formulating the boundary conditions.

Another example of the development of stresses in a body due to the action of mass forces occurs in celestial mechanics. When a meteorite travels through the atmosphere, it is slowed down due to air resistance. At the same time, a certain gas pressure distribution arises on the surface of the meteorite which must be taken into account when formulating the boundary conditions. Because of the complexity of the problem of describing the stress distribution in an actual meteorite, we will consider the idealized problem of the stress-state of an infinitely long circular cylinder of radius R, the longitudinal axis of which is perpendicular to its direction of motion.

Hence, a plane strain problem (when the normal deformation is equal to zero) will be considered in both cases. In this case, the components of the mass force density vector have the form $(Cg_1, Cg_2, 0)$, where C is the density of the material, and g_1 and g_2 are components of the acceleration vector.

We now introduce the complex quantity

$$G = g_1+ig_2$$

and specify the potential in the form

$$V = -C(g_1x_1+g_2x_2)$$

It can be shown that

$$Cg_1 = -\frac{\partial V}{\partial x_1}, \quad Cg_2 = -\frac{\partial V}{\partial x_2}$$

In complex form, the potential has the form

$$V = -\frac{C}{2}(G\bar{z}+\overline{G}z) \tag{4.1}$$

and, consequently,

$$W = \frac{1-2\nu}{8(1-\nu)}C(\overline{G}z^2\bar{z}+Gz\bar{z}^2) \tag{4.2}$$

It obviously follows that the conformal mapping

$$z = R\varsigma \tag{4.3}$$

must be used in both cases. Here

$$G_1(\varsigma, \bar{\varsigma}) = \frac{1-2\nu}{4(1-\nu)} CR(G\bar{\varsigma} + \bar{G}\varsigma), \quad H_1(\varsigma, \bar{\varsigma}) = \frac{1-2\nu}{4(1-\nu)} CR\bar{G}\bar{\varsigma}$$

and

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} = 2[\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{1}{2(1-\nu)} CR(G\bar{\varsigma} + \bar{G}\varsigma) \tag{4.4}$$

$$\sigma_{\vartheta\vartheta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\vartheta} = 2\frac{\varsigma^2}{\rho^2}\left\{[\bar{\varsigma}\Phi_1'(\varsigma) + \Psi_1(\varsigma)] + \frac{1-2\nu}{4(1-\nu)} CR\bar{G}\bar{\varsigma}\right\} \tag{4.5}$$

The combination of components of the stress tensor used in specifying the boundary conditions takes the form

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta} = [\Phi_1(\varsigma) + \overline{\Phi_1(\varsigma)}] - \frac{\varsigma^2}{\rho^2}[\bar{\varsigma}\Phi_1'(\varsigma) + \Psi_1(\varsigma)] -$$
$$- \frac{1}{4(1-\nu)} CR(G\bar{\varsigma} + \bar{G}\varsigma) - \frac{\varsigma^2}{\rho^2}\left[\frac{1-2\nu}{4(1-\nu)} CR\bar{G}\bar{\varsigma}\right] \tag{4.6}$$

4.1. Stresses in a circular cylinder moving in a resisting medium

We shall assume that an infinitely long cylinder of radius R, moving in a transverse direction in a resisting medium, is decelerated by the action of this medium. Due to the resistance of the medium, we shall assume a certain pressure distribution on the surface of the cylinder to be given. The action of the pressure forces is compensated by inertial forces arising as a consequence of the deceleration.

We shall assume that the direction of motion is given as being along the OX_1 coordinate axis which, like the OX_2 axis, that is orthogonal to it, is located in a cross-section plane. In the problem considered, it is convenient in some cases to use the cylindrical system of coordinates (r, α, x_3).

To be specific, we will take the components of the stress tensor on the cylinder surface in the form

$$r = R, \quad -\alpha_0 \le \alpha \le \alpha_0: \quad \sigma_{rr} = -p\cos\alpha, \quad \sigma_{r\alpha} = g\sin 2\alpha$$

or, in the complex form,

$$r = R, \quad -\alpha_0 \le \alpha \le \alpha_0: \quad \sigma_{rr} = -\frac{p}{2R}(z + \bar{z}), \quad \sigma_{r\alpha} = \frac{q}{2iR^2}(z^2 - \bar{z}^2)$$

Starting from the boundary conditions which have been adopted, we write the expression for the boundary condition (4.6), taking account of the conformal mapping (4.3), in the form

$$[\Phi_1(t) + \overline{\Phi_1(t)}] - t^2[t^{-1}\Phi_1'(t) + \Psi_1(t)] -$$
$$- \frac{1}{4(1-\nu)} CR(Gt^{-1} + \bar{G}t) - \frac{1-2\nu}{4(1-\nu)} CR\bar{G}t = -p\cos\vartheta - iq\sin 2\vartheta \tag{4.7}$$

where p and q are constants and $t = e^{i\upsilon}$ is an arbitrary point of the contour γ: $|\rho| = 1$, $-\alpha_0 \le \upsilon \le \alpha_0$.

We shall seek the solution of the problem using complex Fourier series. To do this, we define

$$\Phi_1(\varsigma) = \sum_{k=0}^{\infty} a_k\varsigma^k, \quad \Psi_1(\varsigma) = \sum_{k=0}^{\infty} b_k\varsigma^k \tag{4.8}$$

In addition to this, we must represent the right-hand side of relation (4.7) in the form of the expansions

$$\cos\vartheta = \sum_{n=-\infty}^{\infty} C_n e^{in\vartheta}, \quad \sin 2\vartheta = \sum_{n=-\infty}^{\infty} D_n e^{in\vartheta}$$

Carrying out this operation using the relations

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} \cos\vartheta e^{-in\vartheta} d\vartheta = \frac{1}{2\pi}\int_{-\alpha_0}^{\alpha_0} \cos\vartheta e^{-in\vartheta} d\vartheta$$

$$D_n = \frac{1}{2\pi}\int_0^{2\pi} \sin 2\vartheta e^{-in\vartheta} d\vartheta = \frac{1}{2\pi}\int_{-\alpha_0}^{\alpha_0} \sin 2\vartheta e^{-in\vartheta} d\vartheta$$

and introducing the notation

$$S_{0,0} = \frac{1}{2\pi}\sin\alpha_0, \quad S^{\pm}_{0,k} = \frac{1}{2\pi}\left(\alpha_0 \pm \frac{\sin k\alpha_0}{k}\right),$$
$$S^{\pm}_{k,l} = \frac{1}{2\pi}\left(\frac{\sin k\alpha_0}{k} \pm \frac{\sin l\alpha_0}{l}\right), \quad k, l = 1, 2, \dots$$

we have

$$C_0 = 2S_{0,0}, \quad C_{-1} = C_1 = S_{0,2}, \quad C_n = S^{+}_{n-1,n+1}, \quad n = \pm 2, \pm 3, \pm 4, \dots$$
$$D_{-2} = -D_2 = iS^{-}_{0,4}, \quad D_n = -iS^{-}_{n-2,n+2}, \quad n \neq \pm 2 \tag{4.9}$$

In order to determine the unknown coefficients in series (4.8), we substitute these expansions and the coefficients (4.9) into boundary condition (4.7). As a result, we obtain

$$\sum_{k=0}^{\infty} a_k e^{ik\vartheta} + \sum_{k=0}^{\infty} \bar{a}_k e^{-ik\vartheta} - \sum_{k=1}^{\infty} k a_k e^{ik\vartheta} - \sum_{k=0}^{\infty} b_k e^{i(k+2)\vartheta} - \frac{1}{4(1-\nu)} CR(Ge^{-i\vartheta} + \bar{G}e^{i\vartheta}) -$$
$$-\frac{1-2\nu}{4(1-\nu)} CR\bar{G}e^{i\vartheta} = -p\sum_{n=-\infty}^{\infty} C_n e^{in\vartheta} - iq\sum_{n=-\infty}^{\infty} D_n e^{in\vartheta} \tag{4.10}$$

Now, by selecting the coefficients of like powers of the exponent, we obtain an infinite system of equations in the unknown coefficients a_k, b_k. It is important to note that one of these equations, namely, the one for the first power of the exponent (that is, when $k = 1$), does not contain the required coefficients of expansions (4.8) and (4.9). It has the form

$$\frac{1}{2} CR\bar{G} = pC_1 + iqD_1$$

or, when the expressions obtained above are used for the quantities appearing in it,

$$CR(g_1 - ig_2) = 2pS^{+}_{0,2} + 2qS^{-}_{1,3} \tag{4.11}$$

This relation expresses the condition for the inertial forces acting on the cylinder, on the one hand and the forces due to the gas pressure on the cylinder, on the other hand, to be equal. We also note that the system of forces acting on the cylinder excludes its rotation.

From relation (4.11), we find

$$g_1 = \frac{2(pS^{+}_{0,2} + qS^{-}_{1,3})}{CR}, \quad g_2 = 0$$

in the case of the specified law for the pressure distribution on the cylinder surface.

In order to determine the unknown coefficients $a_0, a_1, a_2, \dots$ successively, we equate the coefficients on the left-hand and right-hand sides of relation (4.10) when $k = 0, -1, -2, \dots$. As a result, we obtain

$$a_0 + \bar{a}_0 = -pC_1 - iqD_0, \quad \bar{a}_1 - \frac{CRG}{4(1-\nu)} = -pC_{-1} - iqD_{-1},$$
$$\bar{a}_2 = -pC_{-2} - iqD_{-2}, \dots, \quad \bar{a}_k = -pC_{-k} - iqD_{-k}, \dots$$

From the first equation, we obtain the real part a_{01} of the coefficient a_0. Note that the imaginary part of this coefficient is not required when determining the components of the stress tensor and it can therefore be put equal to zero. From the next equations, we find

$$a_1 = \frac{CRG}{4(1-\nu)} - p\bar{C}_{-1} - iq\bar{D}_{-1}, \quad a_k = -p\bar{C}_{-k} - iq\bar{D}_{-k}, \quad k \geq 2$$

In order to determine the coefficients $b_0, b_1, b_2, \dots$, we collect all the terms for $e^{ik\upsilon}$, $k \geq 2$. As a result, we arrive at the system of equations

$$a_2 - 2a_2 - b_0 = -pC_2 - iqD_2, \quad a_3 - 3a_3 - b_1 = -pC_3 - iqD_3, \dots$$
$$\dots, a_k - ka_k - b_{k-2} = -pC_k - iqD_k, \dots$$

Then, we find

$$b_k = pC_{k+2} + iqD_{k+2} - (k+1)a_{k+2}, \quad k \geq 0$$

The coefficients obtained enable us to find the form of the complex potentials (4.8). Omitting all the intermediate operations, we present them taking account of relation $\zeta = z/R$:

$$\Phi(z) = -pS_{0,0} + \left[\frac{CRG}{4(1-\nu)} - pS_{0,2}^{+} + qS_{1,3}^{-}\right]\frac{z}{R} +$$

$$+[-pS_{1,3}^{+} + qS_{0,4}^{-}]\frac{z^2}{R^2} - \sum_{k=3}^{\infty}(pS_{k+1,k-1}^{+} + qS_{k+2,k-2}^{+})\left(\frac{z}{R}\right)^k$$

$$\Psi(z) = p\sum_{k=0}^{\infty}(k+2)S_{k+1,k+3}^{+}\left(\frac{z}{R}\right)^k - q\sum_{k=1}^{\infty}kS_{k,k+4}^{-}\left(\frac{z}{R}\right)^k$$

The main combinations for the components of the stress tensor are given by the relations

$$\sigma_{rr} + \sigma_{\alpha\alpha} = 2[\Phi(z) + \overline{\Phi(z)}] - \frac{1}{2(1-\nu)}C(G\bar{z} + \bar{G}z) \tag{4.12}$$

$$\sigma_{\alpha\alpha} - \sigma_{rr} + 2i\sigma_{r\alpha} = 2e^{2i\alpha}\left\{[\bar{z}\Phi'(z) + \Psi(z)] + \frac{1-2\nu}{4(1-\nu)}C\bar{G}\bar{z}\right\} \tag{4.13}$$

Then, on separating the real and imaginary parts, we find the individual components of the stress tensor.

4.2. Stresses in a circular cylinder lying on a smooth plane

We will assume that an infinitely long circular cylinder of radius R lies on a smooth surface and that the width of the strip where the cylinder and surface come into contact is finite but much narrower than R, and the reactive force of the base, which balances the weight of the cylinder, is uniformly distributed over the contact area.

We now introduce a rectangular system of Cartesian coordinates in the following manner. We consider a certain cross-section of the cylinder and we place the OX_1 and OX_2 axes in the plane of this section. The origin of coordinates is placed at the centre of the circular section of the cylinder. We shall consider the OX_1 axis as passing through the centre of the contour of the contact area.

In the cylindrical system of coordinates (r, α, x_3), the boundary conditions on the cylinder surface taken the form

$$r = R:\ \sigma_{rr} = \begin{cases} -p, & -\alpha_0 \le \alpha \le \alpha_0 \\ 0, & \alpha_0 < \alpha < 2\pi - \alpha_0; \end{cases} \qquad \sigma_{r\alpha} = 0, \quad 0 \le \alpha \le 2\pi$$

When the expression for the boundary condition (4.6) is used and account is taken of the conformal mapping (4.3), we have

$$\Phi_1(t) + \overline{\Phi_1(t)} - t\Phi_1'(t) - t^2\Psi_1(t) - \frac{CRGt^{-1}}{4(1-\nu)} - \frac{CR\bar{G}t}{2} = -p \tag{4.14}$$

where p is a constant and $t = e^{i\upsilon}$ is an arbitrary point of the contour γ: $|\rho| = 1$, $-\alpha_0 \le \upsilon \le \alpha_0$. In the case in question, we shall assume that $G = g_1$.

We will seek the complex potentials in the form of (4.8). At the same time, we will represent the right-hand side of the boundary condition (4.14) by a Fourier series in the complex domain in the form

$$-p = \sum_{n=-\infty}^{\infty} C_n e^{in\vartheta}$$

The complex Fourier coefficients are determined from the formula

$$C_n = -\frac{p}{2\pi}\int_{-\alpha_0}^{\alpha_0} e^{-in\vartheta}d\vartheta$$

Evaluating the last integral, we obtain

$$C_0 = -\frac{p\alpha_0}{\pi}, \quad C_k = -\frac{p}{k\pi}\sin k\alpha_0, \quad k = \pm 1, \pm 2, \pm 3, \ldots$$

Direct use of the above-mentioned expansions in boundary conditions (4.14) gives the relation

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} \bar{a}_k t^{-k} - \sum_{k=1}^{\infty} k a_k t^k - \sum_{k=0}^{\infty} b_k t^{k+2} - \frac{CRGt^{-1}}{4(1-\nu)} - \frac{CR\bar{G}t}{2} = \sum_{n=-\infty}^{\infty} C_n t^n$$

In order to determine the unknown coefficients a_k, b_k, we equate the coefficients of like powers of t on both sides of the last equality in the following order: initially when $t^0, t^{-1}, t^{-2}, \ldots$ and, then, when $t^2, t^3, \ldots$. As a result, we obtain

$$a_0 = -\frac{p\alpha_0}{2\pi}, \quad a_1 = \frac{CRG}{4(1-\nu)} - \frac{p}{\pi}\sin\alpha_0, \quad a_k = -\frac{p}{\pi}\sin k\alpha_0, \quad k = 2, 3, 4, \ldots$$

$$b_k = \frac{p}{(k+1)\pi}[1-(k+1)\sin(k+2)\alpha_0], \quad k = 0, 1, 2, \ldots$$

It is especially necessary to separate out the case when the coefficients of t are equal. In this case, we arrive at the relation

$$-\frac{CR\bar{G}}{2} = C_1 = -\frac{p}{\pi}\sin\alpha_0$$

Using this equation, we obtain the value of the pressure which is uniformly distributed over the contact area between the cylinder and the base of unit length

$$p = \frac{\pi CRg_1}{2\sin\alpha_0}$$

Note that this formula can be obtained from the condition that the main vector of the uniformly distributed contact forces and the weight of the cylinder are equal.

Hence, the expressions for the required complex potentials, taking account of the relation $\zeta = r/R$, have the form

$$\Phi(z) = -\frac{p\alpha_0}{2\pi} + \frac{CRG}{4(1-\nu)}\frac{z}{R} - \frac{p}{\pi}\sum_{k=1}^{\infty}\left(\frac{z}{R}\right)^k \sin k\alpha_0$$

$$\Psi(z) = \frac{p}{\pi}\sum_{k=0}^{\infty}\left(\frac{z}{R}\right)^k \frac{1-(k+1)\sin(k+2)\alpha_0}{k+1}$$

In the same way as above, the components of the stress tensor are found using relations (4.12) and (4.13) by substituting the expressions obtained for the complex potentials into them and separating the real and imaginary parts from them with subsequent separation of the normal components of the stress tensor.

It can be established by direct verification that the solutions obtained guarantee that the specified boundary conditions are satisfied in all the problems which have been presented.

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